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Inductance calculation of polygonal conductors

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Abstract
The analytical expressions of the self- and mutual inductances of polygonal conductors with a uniform current density over their surfaces or cross sections are obtained from the calculation of the geometrical mean distances due to the complex variable method, as an extension of the previous results for the circular and rectangular conductors.

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1. Introduction

The inductance calculations for various conductors have been developed since Maxwell [1–4]. However, the analytical expressions of inductances have not been obtained for the polygonal conductors including the triangular conductors with a uniform current density over the cross section. On the other hand, the complex variable methods for computing two-dimensional (2D) magnetic fields have been extended mainly by Beth [5–8] and Halbach [9]. Complex variable methods have been used as a convenient formalism and a powerful mathematical technique for the 2D magnetic field problems [10–12]. Furthermore, the analytical rigorous expressions of the magnetic fields and the vector potential produced by infinitely long straight polygonal and pie-shaped conductors with a uniform current density have been obtained [13, 14]. The purpose of this paper is to obtain the analytical expressions of the geometrical mean distances related to the self- and mutual inductances for polygonal conductors with a uniform current density over their surfaces or cross sections, using the complex variable methods.

2. Inductance expression

2.1. Inductances for straight conductors

The self- and mutual inductances for two parallel straight conductors of the length l with the arbitrary conductor boundary, as shown in figure 1, can be expressed as follows [1–4]:

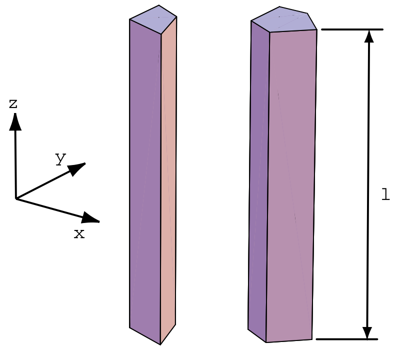

Figure 1. Schematic view of two parallel polygonal conductors.

$$L_{pq} = \frac{1}{I_p I_q} \int \vec{j}_q \cdot \vec{A}_p \, \mathrm{d}V_q = \frac{\mu_0}{4\pi} \frac{1}{I_p I_q} \int\int \vec{j}_q \cdot \vec{j}_p \frac{\mathrm{d}V_p \, \mathrm{d}V_q}{r_{pq}}$$
$$= \frac{\mu_0}{4\pi} \frac{1}{S_p S_q} \int \mathrm{d}S_p \int \mathrm{d}S_q \int_0^l \int_0^l \frac{\vec{\mathrm{d}h}_p \cdot \vec{\mathrm{d}h}_q}{r_{pq}} \tag{1}$$

where I_p and I_q are the currents for each conductor, j_{p} and j_q are the current densities, A_p is the vector potential due to pth conductor, S_p and S_q are the cross-sectional areas and h_p and h_q are the axial lengths. In this paper, parallel straight conductors with the z-directional surface or volume current perpendicular to the xy plane, as shown in figure 1 are only considered. In particular, on the limiting case for $l \gg r$, where r is the distance between two parallel filaments within each conductor, the integration along two filaments with the same z-directional current can be expressed as follows [1–4]:

$$\lim_{r/l \to 0} \left[\frac{\mu_0}{4\pi} \int_0^l \int_0^l \frac{\vec{\mathrm{d}l}_p \cdot \vec{\mathrm{d}l}_q}{r_{pq}} \right] = \frac{\mu_0 l}{2\pi} \left(\ln \frac{2l}{r} - 1 \right). \tag{2}$$

Then, the self- and mutual inductances for two parallel conductors can be expressed, using the geometrical mean distance R_{pq} as follows [1–4]:

$$L_{pq} = \frac{\mu_0 l}{2\pi} (\ln 2l - 1) - \frac{\mu_0 l}{2\pi} \frac{1}{S_p S_q} \int\int \ln r \, \mathrm{d}S_p \, \mathrm{d}S_q = \frac{\mu_0 l}{2\pi} (\ln 2l - 1) - \frac{\mu_0 l}{2\pi} \ln R_{pq}. \tag{3}$$

2.2. Geometrical mean distances for arbitrary current sheets and conductors

The geometrical mean distance R_{pq} for the arbitrary current sheets of an infinitesimal thickness can be expressed in the complex form with the cross-sectional lines, instead of the cross-sectional areas S_p and S_q in equation (3) as follows:

$$\ln R_{pq} = \frac{1}{l_p l_q} \int\int \ln r \, \mathrm{d}l_p \, \mathrm{d}l_q = \frac{1}{l_p l_q} \int_{z \in l_q} \int_{\zeta \in l_p} \frac{\ln(z - \zeta) + \ln(z^* - \zeta^*)}{2} \, \mathrm{d}l_p \, \mathrm{d}l_q \tag{4}$$

where l_p and l_q are the lengths of the cross-sectional lines of two current sheets for the surface current.

The geometrical mean distance R_{pq} for conductors of the arbitrary cross section can be expressed, transforming from the surface integral to the contour integral with the counterclockwise direction as follows [9]:

$$\ln R_{pq} = \frac{1}{S_p S_q}\int\!\!\int \ln r \,\mathrm{d}S_p\,\mathrm{d}S_q = \frac{1}{S_p S_q}\int_{z\in S_q}\int_{\zeta\in S_p}\frac{\ln(z-\zeta)+\ln(z^*-\zeta^*)}{2}\,\mathrm{d}S_p\,\mathrm{d}S_q$$
$$= \frac{\mathrm{i}}{4}\frac{1}{S_p S_q}\int_{z\in S_q}\oint (z^*-\zeta^*)\{\ln(z-\zeta)+\ln(z^*-\zeta^*)-1\}\,\mathrm{d}\zeta\,\mathrm{d}S_q$$
$$= -\frac{1}{8}\frac{1}{S_p S_q}\oint\oint (z^*-\zeta^*)(z-\zeta)\{\ln(z-\zeta)+\ln(z^*-\zeta^*)-2\}\,\mathrm{d}\zeta\,\mathrm{d}z^* \quad (5)$$

where the following differential relations are used on the transformation from the surface integral to the contour integral due to Stokes' theorem [9, 15]:

$$\frac{\partial}{\partial\zeta^*}[(z^*-\zeta^*)\{\ln(z-\zeta)+\ln(z^*-\zeta^*)-1\}]$$
$$= -\{\ln(z-\zeta)+\ln(z^*-\zeta^*)-1\}+(z^*-\zeta^*)\frac{-1}{(z^*-\zeta^*)}$$
$$= -\{\ln(z-\zeta)+\ln(z^*-\zeta^*)\} \quad (6)$$

$$\frac{\partial}{\partial z}[(z^*-\zeta^*)(z-\zeta)\{\ln(z-\zeta)+\ln(z^*-\zeta^*)-2\}]$$
$$= (z^*-\zeta^*)\{\ln(z-\zeta)+\ln(z^*-\zeta^*)-2\}+(z^*-\zeta^*)(z-\zeta)\frac{1}{(z-\zeta)}$$
$$= (z^*-\zeta^*)\{\ln(z-\zeta)+\ln(z^*-\zeta^*)-1\}. \quad (7)$$

In addition, it can be easily verified that the geometrical mean distance R_{pq}, given by equation (5), is a real quantity, as seen from the fact that the complex conjugate of equation (5) with the interchange of the dummy integral variables of ζ and z is identical to the original form of equation (5).

3. Circular conductor

3.1. Circular sheets

In particular, the complex forms of equations (4) and (5) for the geometrical mean distance R_{pq} can easily be evaluated for round conductors with a uniform current density over their surfaces or cross sections. In this section, the calculation of the geometrical mean distance R_{pq} for round conductors is firstly presented, in order to show that complex variable methods are the effective mathematical technique, compared to the conventional method. At first, the geometrical mean distances of circular conductors with the uniform surface current perpendicular to the xy plane shown in figure 2 are calculated. The first contour integration with respect to ζ of equation (4) for the geometrical mean distance R_{pq} of the circular current sheet with a radius a, whose centre ζ_0 (= 0) is located at the origin as shown in figure 2, can be easily made for the exterior ($|z| > |\zeta|$, $r > a$), using the residue theorem as follows:

$$\oint\frac{\ln(z-\zeta)+\ln(z^*-\zeta^*)}{2}\,\mathrm{d}l_p = \oint\frac{\ln(z-\zeta)+\ln(z^*-\zeta^*)}{2}a\,\mathrm{d}\varphi$$
$$= \oint\frac{\ln(z-\zeta)+\ln(z^*-\zeta^*)}{2}\left(-\mathrm{i}\frac{a}{\zeta}\right)\mathrm{d}\zeta$$
$$= -\mathrm{i}a\oint\frac{1}{\zeta}\left\{\ln|z|-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left(\frac{\zeta}{z}\right)^n+\left(\frac{\zeta^*}{z^*}\right)^n\right\}\right\}\mathrm{d}\zeta$$
$$= -\mathrm{i}a\ln|z|\times 2\pi\mathrm{i} = 2\pi a\ln|z|. \quad (8)$$

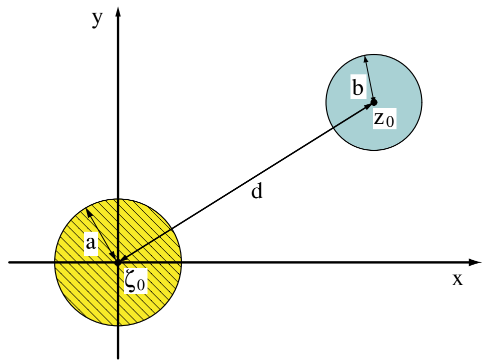

Figure 2. Cross section of two circular conductors.

Then, the geometrical mean distance R_{pq} between two circular current sheets can be also obtained due to the sequential second contour integration with respect to z for a circle with the radius b, located at z_0, as shown in figure 2 as follows:

$$\begin{aligned}\ln R_{pq} &= \frac{1}{(2\pi a)(2\pi b)}\oint 2\pi a \ln|z|\,\mathrm{d}l_q = \frac{1}{2\pi b}\oint \ln|z' + z_0| b\,\mathrm{d}\theta \\ &= \frac{1}{2\pi b}\oint \ln|z' + z_0| \left(-\mathrm{i}\frac{b}{z'}\right)\mathrm{d}z' \\ &= \frac{-\mathrm{i}}{2\pi}\oint \frac{1}{z'}\left\{\ln|z_0| - \sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\left\{\left(\frac{z'}{z_0}\right)^n + \left(\frac{z'^*}{z_0^*}\right)^n\right\}\right\}\mathrm{d}z' \\ &= \frac{-1}{2\pi}\ln d \times 2\pi\mathrm{i} = \ln d \end{aligned} \tag{9}$$

where $2\pi a$ and $2\pi b$ are the circular peripheral lengths of two circular conductors for l_p and l_q in equation (4), respectively. Similarly, from the fact that the resultant expression of equation (8) is correct even at the boundary circle of $|z| = a$, the well-known expression for the geometrical mean distance R_{pp} of a circular current sheet from itself can be obtained as follows:

$$\begin{aligned}\ln R_{pp} &= \frac{1}{(2\pi a)^2}\oint 2\pi a \ln a\,\mathrm{d}l_p = \frac{1}{2\pi a}\oint \ln a \cdot a\,\mathrm{d}\theta \\ &= \frac{1}{2\pi a}\oint \ln a\left(-\mathrm{i}\frac{a}{z}\right)\mathrm{d}z = \frac{-\mathrm{i}\ln a}{2\pi} \times 2\pi\mathrm{i} = \ln a. \end{aligned} \tag{10}$$

3.2. Circular conductors

The first contour integration with respect to ζ of equation (5) for the geometrical mean distance R_{pq} between two parallel round conductors, as shown in figure 2, can be easily made for the exterior ($|z| > |\zeta|, r > a$), using the residue theorem as follows:

$$\begin{aligned}&\oint (z^* - \zeta^*)(z - \zeta)\{\ln(z - \zeta) + \ln(z^* - \zeta^*) - 2\}\,\mathrm{d}\zeta \\ &\qquad = 2\oint \left(z^* - \frac{a^2}{\zeta}\right)(z - \zeta)\left\{\ln|z| - 1 - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\left(\frac{\zeta}{z}\right)^n + \left(\frac{a^2}{z^*\zeta}\right)^n\right\}\right\}\mathrm{d}\zeta\end{aligned}$$

$$= 2\oint \frac{1}{\zeta}\left\{-(z^*z + a^2)\frac{a^2}{2z^*} + \frac{a^4}{4z^*} - a^2 z(\ln|z| - 1)\right\} d\zeta$$
$$= 2\left\{\left(\frac{1}{2} - \ln|z|\right)a^2 z - \frac{a^4}{4z^*}\right\} 2\pi \mathrm{i}. \qquad (11)$$

Finally, the well-known geometrical mean distance R_{pq} between two parallel round conductors can be also obtained due to the sequential second contour integration with respect to z^* for a round conductor with the radius b, located at z_0, as shown in figure 2 as follows:

$$\ln R_{pq} = -\frac{\mathrm{i}4\pi}{8S_pS_q}\oint\left\{\left(\frac{1}{2} - \ln|z|\right)a^2 z - \frac{a^4}{4z^*}\right\} dz^*$$
$$= \frac{\pi}{\mathrm{i}2\pi^2 a^2 b^2}\oint\left\{\left(\frac{1}{2} - \ln|z' + z_0|\right)a^2(z' + z_0) - \frac{a^4}{4(z' + z_0)^*}\right\} dz'^*$$
$$= \frac{1}{\mathrm{i}2\pi b^2}\oint\left\{\left(\frac{1}{2} - \ln|z_0| + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\left\{\left(\frac{z'}{z_0}\right)^n + \left(\frac{z'^*}{z_0^*}\right)\right\}^n\right)\right.$$
$$\left.\times\left(\frac{b^2}{z'^*} + z_0\right) - \frac{a^2}{4z_0^*}\sum_{n=0}^{\infty}\left(\frac{z'^*}{z_0^*}\right)^n\right\} dz'^*$$
$$= \frac{1}{\mathrm{i}2\pi b^2}\oint\frac{b^2}{z'^*}\left\{\left(\frac{1}{2} - \ln d\right) - \frac{1}{2}\right\} dz'^* = \frac{1}{\mathrm{i}2\pi}(-\ln d)\,(-2\pi\mathrm{i}) = \ln d. \qquad (12)$$

As a result, the well-known expression for the geometrical mean distance R_{pp} of the round conductor from itself can be also obtained, due to the residue theorem for the sequential second contour integration with respect to z^* as follows:

$$\ln R_{pp} = -\frac{\mathrm{i}4\pi}{8S_pS_p}\oint\left\{\left(\frac{1}{2} - \ln a\right)a^2 z - \frac{a^4}{4z^*}\right\} dz^*$$
$$= -\frac{\mathrm{i}4\pi}{8(\pi a^2)^2}\oint\frac{a^4}{z^*}\left\{\left(\frac{1}{2} - \ln a\right) - \frac{1}{4}\right\} dz^*$$
$$= -\frac{\mathrm{i}}{2\pi}\left(\frac{1}{4} - \ln a\right)(-2\pi\mathrm{i}) = \ln a - \frac{1}{4} = \ln(a\,\mathrm{e}^{-1/4}). \qquad (13)$$

4. Polygonal conductor

4.1. Polygonal sheets

The first contour integration with respect to ζ of equation (4) for the geometrical mean distance R_{pq} between an N-sided polygonal sheet conductor of the vertices, $\zeta_1, \zeta_2, \ldots,$ and ζ_N with $\zeta_{N+1} = \zeta_1$, and an M-sided polygonal sheet conductor of the vertices, $z_1, z_2, \ldots,$ and z_M with $z_{M+1} = z_1$, as shown in figures 1 and 3, can be made as follows [13]:

$$\ln R_{pq} = \frac{1}{l_p l_q}\int_{z\in l_q}\int_{\zeta\in l_p}\frac{\ln(z-\zeta) + \ln(z^* - \zeta^*)}{2}\,dl_p\,dl_q$$
$$= \frac{1}{l_p l_q}\sum_{j=1}^{M}\sum_{k=1}^{N}\int_{z_j}^{z_{j+1}}\int_{\zeta_k}^{\zeta_{k+1}}\frac{\ln(z-\zeta) + \ln(z^* - \zeta^*)}{2}\frac{|\Delta\zeta_k|}{\Delta\zeta_k}\frac{|\Delta z_j|}{\Delta z_j}\,d\zeta\,dz \qquad (14)$$

where the following relations of the transformation from a segmental line to its complex expression from ζ_k to ζ_{k+1} are used for each cross-sectional line:

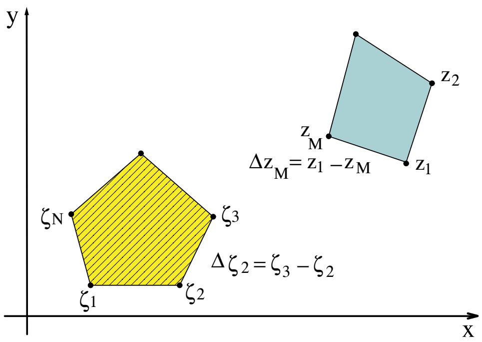

Figure 3. Cross section of two polygonal conductors.

$$l_p = \sum_{k=1}^{N} |\zeta_{k+1} - \zeta_k| \qquad \mathrm{d}l_p = \frac{|\zeta_{k+1} - \zeta_k|}{\zeta_{k+1} - \zeta_k}\,\mathrm{d}\zeta = \frac{|\Delta\zeta_k|}{\Delta\zeta_k}\,\mathrm{d}\zeta, \qquad \text{for} \quad \zeta_k \leqslant \zeta \leqslant \zeta_{k+1}. \tag{15}$$

Then

$$\begin{aligned}
&\int_{\zeta_k}^{\zeta_{k+1}} \frac{\ln(z-\zeta) + \ln(z^*-\zeta^*)}{2} \frac{|\Delta\zeta_k|}{\Delta\zeta_k}\,\mathrm{d}\zeta \\
&\quad = \frac{1}{2}\frac{|\Delta\zeta_k|}{\Delta\zeta_k} \left\{ -[\zeta' \ln\zeta' - \zeta']_{\zeta'=z-\zeta_k}^{\zeta'=z-\zeta_{k+1}} - \frac{\Delta\zeta_k}{\Delta\zeta_k^*} [\zeta' \ln\zeta' - \zeta']_{\zeta'=z^*-\zeta_k^*}^{\zeta'=z^*-\zeta_{k+1}^*} \right\} \\
&\quad = -\frac{1}{2}\frac{|\Delta\zeta_k|}{\Delta\zeta_k} \left[(z-\zeta)\ln(z-\zeta) - (z-\zeta) + \frac{\Delta\zeta_k}{\Delta\zeta_k^*}\{(z^*-\zeta^*)\ln(z^*-\zeta^*) - (z^*-\zeta^*)\} \right]_{\zeta_k}^{\zeta_{k+1}} \\
&\quad = -\frac{1}{2}|\Delta\zeta_k| \left[\frac{(z-\zeta)\ln(z-\zeta) - (z-\zeta)}{\Delta\zeta_k} + \frac{(z^*-\zeta^*)\ln(z^*-\zeta^*) - (z^*-\zeta^*)}{\Delta\zeta_k^*} \right]_{\zeta_k}^{\zeta_{k+1}}.
\end{aligned} \tag{16}$$

With the mathematical manipulation, equation (14) is reduced as follows:

$$\begin{aligned}
&\ln R_{pq} \\
&\quad = -\frac{1}{2}\frac{1}{l_p l_q} \sum_{j=1}^{M}\sum_{k=1}^{N} |\Delta\zeta_k| \int_{z_j}^{z_{j+1}} \left[\frac{(z-\zeta)\ln(z-\zeta) - (z-\zeta)}{\Delta\zeta_1} + \text{C.C.} \right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}} \frac{|\Delta z_j|}{\Delta z_j}\,\mathrm{d}\zeta\,\mathrm{d}z \\
&\quad = -\frac{1}{2}\frac{1}{l_p l_q} \sum_{j=1}^{M}\sum_{k=1}^{N} \frac{|\Delta\zeta_k|}{\Delta\zeta_k}\frac{|\Delta z_j|}{\Delta z_j} \int_{z_j}^{z_{j+1}} [(z-\zeta)\ln(z-\zeta) - (z-\zeta) + \text{C.C.}]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\,\mathrm{d}z \\
&\quad = -\frac{1}{2}\frac{1}{l_p l_q} \sum_{j=1}^{M}\sum_{k=1}^{N} \frac{|\Delta\zeta_k|}{\Delta\zeta_k}\frac{|\Delta z_j|}{\Delta z_j} \left[\left[\frac{(z-\zeta)^2}{2}\left\{\ln(z-\zeta) - \frac{1}{2}\right\} - \frac{(z-\zeta)^2}{2} \right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}} \\
&\qquad\quad + \text{C.C.} \\
&\quad = -\frac{1}{l_p l_q} \sum_{j=1}^{M}\sum_{k=1}^{N} \mathrm{Re}\left[\frac{|\Delta\zeta_k|}{\Delta\zeta_k}\frac{|\Delta z_j|}{\Delta z_j} \left[\left[\frac{(z-\zeta)^2}{2}\left\{\ln(z-\zeta) - \frac{3}{2}\right\} \right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}} \right]
\end{aligned} \tag{17}$$

where C.C. means the term of the complex conjugate. Due to the multi-valued property of the complex logarithmic function, the term with the complex logarithmic function must be evaluated as follows [8,13]:

$$
\begin{aligned}
&\left[\left[\frac{(z-\zeta)^2}{2}\left\{\ln(z-\zeta)-\frac{3}{2}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}} \\
&\quad=\left\{\frac{(z_{j+1}-\zeta_{k+1})^2+(z_j-\zeta_{k+1})^2+(z_{j+1}-\zeta_k)^2+(z_j-\zeta_k)^2}{8}\right\}\ln\frac{(z_{j+1}-\zeta_{k+1})}{(z_j-\zeta_{k+1})}\frac{(z_j-\zeta_k)}{(z_{j+1}-\zeta_k)} \\
&\quad=\left\{\frac{(z_{j+1}-\zeta_{k+1})^2+(z_j-\zeta_{k+1})^2-(z_{j+1}-\zeta_k)^2-(z_j-\zeta_k)^2}{8}\right\}\ln\frac{(z_{j+1}-\zeta_{k+1})}{(z_j-\zeta_{k+1})}\frac{(z_{j+1}-\zeta_k)}{(z_j-\zeta_k)} \\
&\quad=\left\{\frac{(z_{j+1}-\zeta_{k+1})^2-(z_j-\zeta_{k+1})^2+(z_{j+1}-\zeta_k)^2-(z_j-\zeta_k)^2}{8}\right\}\ln\frac{(z_{j+1}-\zeta_{k+1})}{(z_{j+1}-\zeta_k)}\frac{(z_j-\zeta_{k+1})}{(z_j-\zeta_k)} \\
&\quad=\left\{\frac{(z_{j+1}-\zeta_{k+1})^2-(z_j-\zeta_{k+1})^2-(z_{j+1}-\zeta_k)^2+(z_j-\zeta_k)^2}{8}\right\} \\
&\qquad\times\ln(z_{j+1}-\zeta_{k+1})(z_j-\zeta_{k+1})(z_{j+1}-\zeta_k)(z_j-\zeta_k) \\
&\qquad-\frac{3}{4}\{(z_{j+1}-\zeta_{k+1})^2-(z_j-\zeta_{k+1})^2-(z_{j+1}-\zeta_k)^2+(z_j-\zeta_k)^2\}.
\end{aligned}
\tag{18}
$$

Furthermore, the geometrical mean distance R_{pp} of a polygonal sheet conductor from itself can be obtained as follows:

$$
\begin{aligned}
&\ln R_{pp} \\
&\quad=-\frac{1}{l_p^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Re}\left[\frac{|\Delta\zeta_k|}{\Delta\zeta_k}\frac{|\Delta\zeta_j|}{\Delta\zeta_j}\lim_{\alpha\to 0}\left\{\left[\left[\frac{(z-\zeta)^2}{2}\left\{\ln(z-\zeta)-\frac{3}{2}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=\zeta_j+\alpha\Delta\zeta_j}^{z=\zeta_{j+1}-\alpha\Delta\zeta_j}\right\}\right] \\
&\quad\cong-\frac{1}{l_p^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{Re}\left[\frac{|\Delta\zeta_k|}{\Delta\zeta_k}\frac{|\Delta\zeta_j|}{\Delta\zeta_j}\left[\left[\frac{(z-\zeta)^2}{2}\left\{\ln(z-\zeta)-\frac{3}{2}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=\zeta_j+\varepsilon\Delta\zeta_j}^{z=\zeta_{j+1}-\varepsilon\Delta\zeta_j}\right]
\end{aligned}
\tag{19}
$$

where $\varepsilon(\ll 1)$ is an arbitrary small number introduced to avoid the indeterminate quantities of $0\times\infty$ in R_{pp}. With an arbitrary small number ε, the geometrical mean distance R_{pp} can be calculated quite precisely. Even for the mutual inductance between two contacted conductors, the similar mathematical treatment with an arbitrary small number ε is effective.

Especially, the geometrical mean distance R_{pp} of a single sheet conductor from ζ_1 to ζ_2 from itself can be obtained consistently with the previous result as follows [1, 2]:

$$
\begin{aligned}
\ln R_{pp}&=-\mathrm{Re}\left[\frac{1}{(\Delta\zeta_1)^2}\left[\left[\frac{(z-\zeta)^2}{2}\left\{\ln(z-\zeta)-\frac{3}{2}\right\}\right]_{\zeta=\zeta_1}^{\zeta=\zeta_2}\right]_{z=\zeta_1}^{z=\zeta_2}\right] \\
&=\mathrm{Re}\left[\frac{\ln(\zeta_2-\zeta_1)(\zeta_1-\zeta_2)}{2}-\frac{3}{2}\right]=\ln|\zeta_2-\zeta_1|-\frac{3}{2}=\ln\left(|\zeta_2-\zeta_1|\times \mathrm{e}^{-\frac{3}{2}}\right).
\end{aligned}
\tag{20}
$$

In particular, it should be notified that the above assumption of the uniform surface current density for polygonal conductors is not physically always appropriate, because the inner magnetic field within polygonal conductors of the uniform surface current is not uniformly zero, differently from that of a single round conductor.

4.2. Polygonal conductors

The first contour integration with respect to ζ of equation (5) for the geometrical mean distance R_{pq} between an N-sided polygonal conductor of the vertices, $\zeta_1, \zeta_2, \ldots$, and ζ_N with

$\zeta_{N+1} = \zeta_1$, and an M-sided polygonal conductor of the vertices, z_1, z_2, ..., and z_M with $z_{M+1} = z_1$, can be made in the counterclockwise order, as shown in figures 1 and 3 as follows [13]:

$$\begin{aligned}
&\oint (z^* - \zeta^*)(z-\zeta)\{\ln(z-\zeta) + \ln(z^*-\zeta^*) - 2\}\,\mathrm{d}\zeta \\
&= \sum_{k=1}^{N}\left[\int_{\zeta_k}^{\zeta_{k+1}} (z^*-\zeta^*)(z-\zeta)\{\ln(z-\zeta) - 2\}\,\mathrm{d}\zeta + \int_{\zeta_k}^{\zeta_{k+1}} (z^*-\zeta^*)(z-\zeta)\ln(z^*-\zeta^*)\,\mathrm{d}\zeta\right] \\
&= -\sum_{k=1}^{N}\left[\left\{z^* - \zeta_k^* - \frac{\Delta\zeta_k^*}{\Delta\zeta_k}(z-\zeta_k)\right\}\left[\frac{(z-\zeta)^2}{2}\left\{\ln(z-\zeta) - \frac{5}{2}\right\}\right]_{\zeta_k}^{\zeta_{k+1}}\right. \\
&\quad + \frac{\Delta\zeta_k^*}{\Delta\zeta_k}\left[\frac{(z-\zeta)^3}{3}\left\{\ln(z-\zeta) - \frac{7}{3}\right\}\right]_{\zeta_k}^{\zeta_{k+1}} \\
&\quad + \frac{\Delta\zeta_k}{\Delta\zeta_k^*}\left\{z - \zeta_k - \frac{\Delta\zeta_k}{\Delta\zeta_k^*}(z-\zeta_k)^*\right\}\left[\frac{(z^*-\zeta^*)^2}{2}\left\{\ln(z^*-\zeta^*) - \frac{1}{2}\right\}\right]_{\zeta_k}^{\zeta_{k+1}} \\
&\quad \left. + \left(\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\right)^2\left[\frac{(z^*-\zeta^*)^3}{3}\left\{\ln(z^*-\zeta^*) - \frac{1}{3}\right\}\right]_{\zeta_k}^{\zeta_{k+1}}\right] \qquad (21)
\end{aligned}$$

where the following relations for the complex conjugate are used for the contour integral along each boundary line of a polygon [9]:

$$\zeta^* = \zeta_k^* + \frac{(\zeta_{k+1}-\zeta_k)^*}{\zeta_{k+1}-\zeta_k}(\zeta-\zeta_k) = \zeta_k^* + \frac{\Delta\zeta_k^*}{\Delta\zeta_k}(\zeta-\zeta_k) = \zeta_k^* + \frac{\Delta\zeta_k^*}{\Delta\zeta_k}(z-\zeta_k) - \frac{\Delta\zeta_k^*}{\Delta\zeta_k}(z-\zeta). \qquad (22)$$

Equation (21) can be reduced as follows:

$$\begin{aligned}
&\oint (z^* - \zeta^*)(z-\zeta)\{\ln(z-\zeta) + \ln(z^*-\zeta^*) - 2\}\,\mathrm{d}\zeta \\
&= -\sum_{k=1}^{N}\left[\left\{z_j^* - \frac{\Delta z_j^*}{\Delta z_j}(z_j - \zeta_{k+1}) + \frac{\Delta z_j^*}{\Delta z_j}(z-\zeta_{k+1}) - \zeta_k^* - \frac{\Delta\zeta_k^*}{\Delta\zeta_k}(z-\zeta_k)\right\}\right. \\
&\quad \times \frac{(z-\zeta_{k+1})^2}{2}\left\{\ln(z-\zeta_{k+1}) - \frac{5}{2}\right\} - \left\{z_j^* - \frac{\Delta z_j^*}{\Delta z_j}(z_j-\zeta_k) + \frac{\Delta z_j^*}{\Delta z_j}(z-\zeta_k)\right. \\
&\quad \left. - \zeta_k^* - \frac{\Delta\zeta_k^*}{\Delta\zeta_k}(z-\zeta_k)\right\} \times \frac{(z-\zeta_k)^2}{2}\left\{\ln(z-\zeta_k) - \frac{5}{2}\right\} \\
&\quad + \frac{\Delta\zeta_k^*}{\Delta\zeta_k}\left\{\frac{(z-\zeta_{k+1})^3}{3}\left\{\ln(z-\zeta_{k+1}) - \frac{7}{3}\right\} - \frac{(z-\zeta_k)^3}{3}\left\{\ln(z-\zeta_k) - \frac{7}{3}\right\}\right\} \\
&\quad + \frac{\Delta\zeta_k}{\Delta\zeta_k^*}\left\{z_j - \frac{\Delta z_j}{\Delta z_j^*}(z_j^* - \zeta_{k+1}^*) + \frac{\Delta z_j}{\Delta z_j^*}(z^* - \zeta_{k+1}^*) - \zeta_k - \frac{\Delta\zeta_k}{\Delta\zeta_k^*}(z^*-\zeta_k^*)\right\} \\
&\quad \times \frac{(z^*-\zeta_{k+1}^*)^2}{2}\left\{\ln(z^*-\zeta_{k+1}^*) - \frac{1}{2}\right\} - \frac{\Delta\zeta_k}{\Delta\zeta_k^*}\left\{z_j - \frac{\Delta z_j}{\Delta z_j^*}(z_j^* - \zeta_k^*)\right. \\
&\quad \left. + \frac{\Delta z_j}{\Delta z_j^*}(z^*-\zeta_k^*) - \zeta_k - \frac{\Delta\zeta_k}{\Delta\zeta_k^*}(z^*-\zeta_k^*)\right\} \times \frac{(z^*-\zeta_k^*)^2}{2}\left\{\ln(z^*-\zeta_k^*) - \frac{1}{2}\right\}
\end{aligned}$$

$$+\left(\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\right)^2\left[\frac{(z^*-\zeta_{k+1}^*)^3}{3}\left\{\ln(z^*-\zeta_{k+1}^*)-\frac{1}{3}\right\}\right.$$
$$\left.\left.-\frac{(z^*-\zeta_k^*)^3}{3}\left\{\ln(z^*-\zeta_k^*)-\frac{1}{3}\right\}\right]\right] \tag{23}$$

where the following relations are used:

$$z^* = z_j^* + \frac{(z_{j+1}-z_j)^*}{z_{j+1}-z_j}(z-z_j) = z_j^* + \frac{\Delta z_j^*}{\Delta z_j}(z-z_j)$$
$$= z_j^* - \frac{\Delta z_j^*}{\Delta z_j}(z_j-\zeta_{k+1}) + \frac{\Delta z_j^*}{\Delta z_j}(z-\zeta_{k+1}) = z_j^* - \frac{\Delta z_j^*}{\Delta z_j}(z_j-\zeta_k) + \frac{\Delta z_j^*}{\Delta z_j}(z-\zeta_k). \tag{24}$$

Then, the geometrical mean distance R_{pq} between two parallel N-sided and M-sided polygonal conductors, shown in figure 3, can be obtained as follows:

$$\ln R_{pq} = \frac{1}{8}\frac{1}{S_pS_q}\sum_{j=1}^{M}\sum_{k=1}^{N} D(\zeta_k,\zeta_{k+1},z_j,z_{j+1}) \tag{25}$$

$$D(\zeta_k,\zeta_{k+1},z_j,z_{j+1})$$
$$= \left\{z_j^*-\zeta_{k+1}^*-\frac{\Delta z_j^*}{\Delta z_j}(z_j-\zeta_{k+1})\right\}\frac{\Delta z_j^*}{\Delta z_j}\left[\frac{(z-\zeta_{k+1})^3}{6}\left\{\ln(z-\zeta_{k+1})-\frac{17}{6}\right\}\right]_{z_j}^{z_{j+1}}$$
$$+\left(\frac{\Delta z_j^*}{\Delta z_j}-\frac{\Delta\zeta_k^*}{\Delta\zeta_k}\right)\frac{\Delta z_j^*}{\Delta z_j}\left[\frac{(z-\zeta_{k+1})^3}{8}\left\{\ln(z-\zeta_{k+1})-\frac{11}{4}\right\}\right]_{z_j}^{z_{j+1}}$$
$$-\left\{z_j^*-\zeta_k^*-\frac{\Delta z_j^*}{\Delta z_j}(z_j-\zeta_k)\right\}\frac{\Delta z_j^*}{\Delta z_j}\left[\frac{(z-\zeta_k)^3}{6}\left\{\ln(z-\zeta_k)-\frac{17}{6}\right\}\right]_{z_j}^{z_{j+1}}$$
$$-\left(\frac{\Delta z_j^*}{\Delta z_j}-\frac{\Delta\zeta_k^*}{\Delta\zeta_k}\right)\frac{\Delta z_j^*}{\Delta z_j}\left[\frac{(z-\zeta_k)^4}{8}\left\{\ln(z-\zeta_k)-\frac{11}{4}\right\}\right]_{z_j}^{z_{j+1}}$$
$$+\frac{\Delta\zeta_k^*}{\Delta\zeta_k}\frac{\Delta z_j^*}{\Delta z_j}\left\{\left[\frac{(z-\zeta_{k+1})^4}{12}\left\{\ln(z-\zeta_{k+1})-\frac{31}{12}\right\}\right]_{z_j}^{z_{j+1}}\right.$$
$$\left.-\left[\frac{(z-\zeta_k)^4}{12}\left\{\ln(z-\zeta_k)-\frac{31}{12}\right\}\right]_{z_j}^{z_{j+1}}\right\}$$
$$+\frac{\Delta\zeta_k^*}{\Delta\zeta_k}\left\{z_j-\zeta_{k+1}-\frac{\Delta z_j}{\Delta z_j^*}(z_j^*-\zeta_{k+1}^*)\right\}\left[\frac{(z^*-\zeta_{k+1}^*)^3}{6}\left\{\ln(z^*-\zeta_{k+1}^*)-\frac{5}{6}\right\}\right]_{z_j}^{z_{j+1}}$$
$$+\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\left(\frac{\Delta z_j}{\Delta z_j^*}-\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\right)\left[\frac{(z^*-\zeta_{k+1}^*)^4}{8}\left\{\ln(z^*-\zeta_{k+1}^*)-\frac{3}{4}\right\}\right]_{z_j}^{z_{j+1}}$$
$$-\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\left\{z_j-\zeta_k-\frac{\Delta z_j}{\Delta z_j^*}(z_j^*-\zeta_k^*)\right\}\left[\frac{(z^*-\zeta_k^*)^3}{6}\left\{\ln(z^*-\zeta_k^*)-\frac{5}{6}\right\}\right]_{z_j}^{z_{j+1}}$$
$$-\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\left(\frac{\Delta z_j}{\Delta z_j^*}-\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\right)\left[\frac{(z^*-\zeta_k^*)^4}{8}\left\{\ln(z^*-\zeta_k^*)-\frac{3}{4}\right\}\right]_{z_j}^{z_{j+1}}$$

$$+\left(\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\right)^2\left\{\left[\frac{(z^*-\zeta_{k+1}^*)^4}{12}\left\{\ln(z^*-\zeta_{k+1}^*)-\frac{7}{12}\right\}\right]_{z_j}^{z_{j+1}}\right.$$
$$\left.-\left[\frac{(z^*-\zeta_k^*)^4}{12}\left\{\ln(z^*-\zeta_k^*)-\frac{7}{12}\right\}\right]_{z_j}^{z_{j+1}}\right\} \tag{26}$$

where S_p and S_q are the cross-sectional areas of N-sided and M-sided polygonal conductors, shown in figure 3, can be calculated, respectively, as follows:

$$S_q=\int_{z\in S_q}\mathrm{d}S_q=-\frac{\mathrm{i}}{2}\oint z^*\mathrm{d}z=-\frac{\mathrm{i}}{2}\sum_{j=1}^{M}\int_{z_j}^{z_{j+1}}\left\{z_j^*+\frac{\Delta z_j^*}{\Delta z_j}(z-z_j)\right\}\mathrm{d}z$$
$$=\sum_{j=1}^{M}\frac{(x_jy_{j+1}-x_{j+1}y_j)}{2}. \tag{27}$$

Finally, due to the following relations:

$$\zeta_k^*-z_j^*-\frac{\Delta z_j^*}{z_j}(\zeta_k-z_j)=\zeta_k^*-z_{j+1}^*+\Delta z_j^*-\frac{\Delta z_j^*}{z_j}(\zeta_k-z_{j+1}+\Delta z_j)$$
$$=\zeta_k^*-z_{j+1}^*-\frac{\Delta z_j^*}{\Delta z_j}(\zeta_k-z_{j+1}). \tag{28}$$

Equation (26) can be reduced to the symmetric form for ζ and z coordinates as follows:

$$D(\zeta_k,\zeta_{k+1},z_j,z_{j+1})$$
$$=\frac{\Delta z_j^*}{\Delta z_j}\left[\left[\frac{(z^*-\zeta^*)(z-\zeta)^3}{6}\left\{\ln(z-\zeta)-\frac{17}{6}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}}$$
$$-\left(\frac{\Delta z_j^*}{\Delta z_j}\right)^2\left[\left[\frac{(z-\zeta)^4}{24}\left\{\ln(z-\zeta)-\frac{37}{12}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}}$$
$$-\frac{\Delta\zeta_k^*}{\Delta\zeta_k}\frac{\Delta z_j^*}{\Delta z_j}\left[\left[\frac{(z-\zeta)^4}{24}\left\{\ln(z-\zeta)-\frac{37}{12}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}}$$
$$+\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\left[\left[\frac{(z-\zeta)(z^*-\zeta^*)^3}{6}\left\{\ln(z^*-\zeta^*)-\frac{5}{6}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}}$$
$$-\left(\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\right)^2\left[\left[\frac{(z^*-\zeta^*)^4}{24}\left\{\ln(z^*-\zeta^*)-\frac{13}{12}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}}$$
$$-\frac{\Delta\zeta_k}{\Delta\zeta_k^*}\frac{\Delta z_j}{\Delta z_j^*}\left[\left[\frac{(z^*-\zeta^*)^4}{24}\left\{\ln(z^*-\zeta^*)-\frac{13}{12}\right\}\right]_{\zeta=\zeta_k}^{\zeta=\zeta_{k+1}}\right]_{z=z_j}^{z=z_{j+1}} \tag{29}$$

where each term with the logarithmic function must be evaluated in the form containing every factor of $(z_{j+1}-\zeta_{k+1})$, $(z_j-\zeta_{k+1})$, $(z_{j+1}-\zeta_k)$ and $(z_j-\zeta_k)$, as shown in equation (18). The geometrical mean distance R_{pq} can be quite accurately calculated, consistently with the previous results for rectangular conductors [2]. Especially, the geometrical mean distance R_{pp} of a polygonal conductor from itself can be obtained as follows:

$$\ln R_{pp} = \frac{1}{8}\frac{1}{S_p^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\lim_{\alpha\to 0}\{D(\zeta_k, \zeta_{k+1}, z_j = \zeta_j + \alpha\Delta\zeta_j, z_{j+1} = \zeta_{j+1} - \alpha\Delta\zeta_j)\}$$

$$\cong \frac{1}{8}\frac{1}{S_p^2}\sum_{j=1}^{N}\sum_{k=1}^{N} D(\zeta_k, \zeta_{k+1}, z_j = \zeta_j + \varepsilon\Delta\zeta_j, z_{j+1} = \zeta_{j+1} - \varepsilon\Delta\zeta_j) \qquad (30)$$

where an arbitrarily small number ε ($\ll 1$) is introduced as well as equation (19). With an arbitrary small number ε, the geometrical mean distance R_{pp} can be quite accurately calculated, consistently with the previous results for rectangular conductors [1, 2].

The complex expressions of the geometrical mean distances R_{pq} and R_{pp} can easily be evaluated, e.g. by *Mathematica*, which can treat the imaginary numbers [16].

5. Conclusion

The analytical expressions of the geometrical mean distances for the self- and mutual inductances of polygonal conductors are obtained as an extension of the previous results for the rectangular conductors with the complex variable method. Then, this calculation method of the self- and mutual inductances is applicable to various 2D electromagnetic problems, from the relations with inductance to the magnetic energy and forces. As a result, it is shown that the complex variable methods transforming from the surface integral to the line or contour integral give a powerful mathematical tool under the unified scheme to obtain the 2D magnetic fields, the vector potentials and the inductances for various conductors.

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